# The fastest way to circle a black hole

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Black-hole spacetimes with a "photonsphere", a hypersurface on which massless particles can orbit the black hole on circular null geodesics, are studied. We prove that among all possible trajectories (both geodesic and non-geodesic) which circle the central black hole, the null circular geodesic is characterized by the *shortest* possible orbital period as measured by asymptotic observers. Thus, null circular geodesics provide the fastest way to circle black holes. In addition, we conjecture the existence of a universal lower bound for orbital periods around compact objects (as measured by flat-space asymptotic observers): $T_\infty \geq 4\pi M$, where $M$ is the mass of the central object. This bound is saturated by the null circular geodesic of the maximally rotating Kerr black hole.



## I. INTRODUCTION

The motion of test particles in black-hole spacetimes has been extensively studied for more than four decades, see [1–4] are references therein. Of particular importance are geodesic motions which provide valuable information on the structure of the spacetime geometry. Circular null geodesics (also known as "photonspheres") are especially interesting from both an astrophysical and theoretical points of view [5]. As pointed out in [4], the optical appearance to external observers of a star undergoing gravitational collapse is related to the properties of the unstable circular null geodesic [4, 6, 7].

Furthermore, null circular geodesics are closely related to the characteristic oscillation modes of black holes (see e.g. [8, 9] for detailed reviews). These quasinormal resonances can be interpreted in terms of null particles trapped at the unstable circular orbit and slowly leaking out [4, 10–14]. The real part of the complex quasinormal frequencies is related to the angular velocity at the unstable null geodesic (as measured by asymptotic observers) while the imaginary part of the resonances is related to the instability timescale of the orbit [4, 10–14] (or the inverse Lyapunov exponent of the geodesic [4]).

An important physical quantity for the analysis of circular orbits in black-hole spacetimes is the angular frequency $\Omega_\infty$ of the orbit as measured by asymptotic observers. In this paper we shall reveal an interesting property of null circular geodesics which is related to this important quantity: We shall show that null circular geodesics provide the fastest way to circle black holes. More explicitly, we shall prove that the null circular geodesic of a black-hole spacetime is characterized by the *shortest* possible orbital period (the largest orbital frequency) as measured by asymptotic observers.

It is worth pointing out that the orbital period $T$ around a spherical compact object must be bounded from below by the mass $M$ of the central object: Suppose the compact object has radius $R$, then obviously $T \geq 2\pi R$. (We shall use natural units in which $G = c = 1$). In addition, the central object must be larger than its gravitational radius, $R \geq 2M$. Thus, the orbital period must be bounded from below by

$$T \geq 4\pi M \ . \tag{1}$$

However, it should be realized that the above reasoning is actually too naive – it does not take into account the possible influence of the spacetime curvature (in the region near the surface of the compact object) on the orbital period. Due to the influence of the gravitational time dilation effect (redshift), the orbital period $T_\infty$ as measured by asymptotic observers would actually be *larger* than $2\pi R$. Moreover, we shall show below that due to the influence of the redshift factor, the circular orbit with the shortest orbital period (as measured by asymptotic observers) is *distinct* from the circular orbit with the smallest circumference (that is, $r_{\text{fast}} \neq R$ in general, where $r_{\text{fast}}$ to be determined below is the radius of the circular trajectory with the shortest orbital period).

## II. SPHERICALLY SYMMETRIC SPACETIMES

We shall first consider static spherically symmetric asymptotically flat black-hole spacetimes. The line element may take the following form in Schwarzschild coordinates [4, 15]

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{2}$$

where the metric functions $f(r)$ and $g(r)$ depend only on the Schwarzschild areal coordinate $r$. These functions should be determined by solving the field equations. Since we do not specify the field equations, our results would be valid for all spherically symmetric asymptotically flat black holes. We note, in particular, that we do *not* assume $g(r) = f(r)$ (a property which characterizes the familiar Schwarzschild and Reissner-Nordström spacetimes) and thus our results would be applicable to hairy black-hole configurations as well [in these spacetimes $g(r) \neq f(r)$, see [16–20] and references therein]. Asymptotic flatness requires that as $r \to \infty$,

$$f(r) \to 1 \quad \text{and} \quad g(r) \to 1 \ , \tag{3}$$

and a regular event horizon at $r = r_H$ requires [21]

$$f(r_H) = g(r_H) = 0 \; . \tag{4}$$

We shall now consider the following question: What is the radius $r = r_{\text{fast}}$ of the circular trajectory [22] around a central black hole which has the shortest orbital period (the largest orbital frequency) as measured by asymptotic observers? It is obvious that in order to minimize the orbital period for a given radius $r$, one should move as close as possible to the speed of light [23]. In this case, the orbital period as measured by asymptotic observers can be obtained from Eq. (2) with $ds = dr = d\theta = 0$ and $\Delta\phi = 2\pi$:

$$T(r) \equiv T_\infty(r) = \frac{2\pi r}{\sqrt{f(r)}} \; , \tag{5}$$

where $r$ is the radius of the circular trajectory. The denominator of (5) represents the well-known redshift factor.

Note that for asymptotically flat spacetimes $T(r) \to \infty$ at both $r \to r_H$ and $r \to \infty$. Thus, the function $T(r)$ must have a minimum for some finite $r = r_{\text{fast}} > r_H$. [In particular, the circular trajectory with the smallest possible radius ($r \to r_H$) is characterized by an infinite orbital period as measured by asymptotic observers. Thus, the flat-space reasoning which leads to $T_{\text{min}} = \min_r\{2\pi r\} = 2\pi R$ for a compact object of radius $R$ is certainly not correct for the curved region near a black hole.]

The circular trajectory around the central black hole which has the shortest orbital period is characterized by

$$T'(r = r_{\text{fast}}) = 0 \; , \tag{6}$$

where a prime denotes a derivative with respect to areal radius $r$. This yields the characteristic equation

$$r_{\text{fast}} = \frac{2f(r_{\text{fast}})}{f'(r_{\text{fast}})} \tag{7}$$

for the circular trajectory with the shortest orbital period.

We shall now prove that it is the null circular geodesic of the black-hole spacetime which has the shortest orbital period as measured by asymptotic observers. To that end, we shall follow the analysis of [2–4] and compute the location $r = r_\gamma$ of the null circular geodesic for a black-hole spacetime described by the line element (2). The Lagrangian describing the geodesics in the spacetime (2) is given by

$$2\mathcal{L} = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\dot{\phi}^2 \; , \tag{8}$$

where a dot denotes a derivative with respect to the affine parameter along the geodesic. The generalized momenta derived from this Lagrangian are given by [2–4]

$$p_t = -f(r)\dot{t} \equiv -E = \text{const} \; , \tag{9}$$

$$p_\phi = r^2\dot{\phi} \equiv L = \text{const} \; , \tag{10}$$

and

$$p_r = \frac{1}{g(r)}\dot{r} \; . \tag{11}$$

The Lagrangian is independent of both $t$ and $\phi$. This implies that $E$ and $L$ are constants of the motion. The Hamiltonian of the system is given by [2–4] $\mathcal{H} = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L}$, which implies

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \frac{1}{g(r)}\dot{r}^2 = \delta = \text{const} \; , \tag{12}$$

where $\delta = 0$ for null geodesics and $\delta = 1$ for timelike geodesics. Substituting Eqs. (9)-(11) into (12), one finds

$$\dot{r}^2 = \frac{g(r)}{f(r)}E^2\Big[1 - b^2\frac{f(r)}{r^2}\Big] \tag{13}$$

for null geodesics, where $b \equiv L/E = \text{const}$.

Circular geodesics are characterized by $\dot{r}^2 = (\dot{r}^2)' = 0$ [2–4]. The requirement $(\dot{r}^2)' = 0$ yields the equation

$$r_\gamma = \frac{2f(r_\gamma)}{f'(r_\gamma)} \tag{14}$$

for the null circular geodesic. Taking cognizance of the fact that Eq. (14) for the null circular geodesic is *identical* to Eq. (7) for the fastest circular trajectory (the circular trajectory with the shortest orbital period), one realizes that the null circular geodesic is characterized by the shortest possible orbital period as measured by asymptotic observers:

$$r_{\text{fast}} = r_\gamma \; . \tag{15}$$

It is worth noting that Eq. (6) [and thus also Eq. (7)] may have several solutions. Nevertheless, our conclusion that the circular trajectory around the black hole with the shortest orbital period coincides with a null circular geodesic of the black-hole spacetime still holds true in such cases as well: The fact that Eq. (14) for the null circular geodesic(s) is identical to Eq. (7) for the circular trajectory(ies) with $T'(r) = 0$, implies that if $T(r)$ has several extremum points [that is, Eq. (7) has several solutions], then the black-hole spacetime would be characterized by several circular null geodesics [that is, Eq. (14) would also have (the same) several solutions]. The main point is that *any* solution of Eq. (7) is *also* a solution of the identical equation (14). Thus, the circular trajectory with the global minimum of $T(r)$ [this trajectory is characterized by the relation $rf' - 2f = 0$, see Eq. (7)] must coincide with (one of) the null circular geodesic(s) of the black-hole spacetime [these null circular geodesics are *also* characterized by the *same* relation, $rf' - 2f = 0$, see Eq. (14)].

The shortest possible orbital period around the black hole as measured by asymptotic observers is obtained by

substituting the optimal radius (15) back into Eq. (5). It can be shown [24] that all spherical black holes (including hairy solutions [16–20]) satisfy the inequality

$$f(r_\gamma) \leq \frac{1}{3} \ . \tag{16}$$

[The case $f(r_\gamma) = 1/3$ corresponds to the ("bare") Schwarzschild black hole.] Taking cognizance of Eqs. (5), (15), and (16), one finds that the shortest possible orbital period around a spherical black hole must conform to the lower bound

$$T_{\text{min}} \equiv \text{min}_r\{T(r)\} \geq 2\sqrt{3}\pi r_\gamma \ . \tag{17}$$

[Compare this with the *weaker* flat-space bound $T \geq 2\pi R$ discussed above.]

## III. ROTATING KERR BLACK HOLES

Our conclusions can be generalized to include non-spherically symmetric black-hole spacetimes. We shall now analyze circular trajectories around rotating Kerr black holes. In Boyer-Lindquist coordinates the line element of the Kerr spacetime takes the form [2, 25]

$$ds^2 = -\Big(1-\frac{2Mr}{\rho^2}\Big)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \Big(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\rho^2}\Big)\sin^2\theta d\phi^2, \tag{18}$$

where $\Delta \equiv r^2-2Mr+a^2$ and $\rho^2 \equiv r^2+a^2\cos^2\theta$. Here $M$ and $a$ are the black-hole mass and angular momentum per unit mass, respectively. The event (outer) horizon of the black hole is located at $r_H = M+(M^2-a^2)^{1/2}$ and the stationary limit surface of the exterior spacetime is located at $r_S = M+(M^2-a^2\cos^2\theta)^{1/2}$.

We shall consider circular orbits in the equatorial plane of the black hole. These are characterized by $\theta = \pi/2$, which implies $\rho = r$ and $r_S = 2M$. Again, it is obvious that in order to minimize the orbital period for a given radius $r$, one should move as close as possible to the speed of light [23]. In this case, the orbital period as measured by asymptotic observers can be obtained from Eq. (18) with $ds = dr = d\theta = 0$ and $\Delta\phi = \pm 2\pi$ [26]:

$$T(r) = 2\pi\frac{\sqrt{\Delta}\mp\frac{2Ma}{r}}{1-\frac{2M}{r}} \ , \tag{19}$$

where the upper/lower signs correspond to co-rotating/counter-rotating orbits. [Note that $T(r \to 2M) \to \infty$ for a counter-rotating orbit as the stationary surface $r = 2M$ is approached.]

The circular trajectory around the central black hole which has the shortest orbital period is characterized by

$$T'(r = r_{\text{fast}}) = 0 \ , \tag{20}$$

which yields the equation

$$r(r-2M)(r-3M) \pm 2Ma(\sqrt{\Delta}\mp a) = 0 \ . \tag{21}$$

It is straightforward to show that a solution of the condition (21) is given by

$$r_{\text{fast}_\pm} = 2M[1+\cos[\frac{2}{3}\cos^{-1}(\mp a/M)]] \ . \tag{22}$$

We shall now prove that the null circular geodesic of the Kerr black hole is characterized by the shortest orbital period as measured by asymptotic observers. To that end, we shall follow the analysis of [2–4] and compute the location $r = r_\gamma$ of the null circular geodesic of the Kerr spacetime. The Lagrangian describing the geodesics in the spacetime (18) is given by

$$2\mathcal{L} = g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2 \ . \tag{23}$$

The generalized momenta derived from this Lagrangian are given by [2–4]

$$p_t = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \equiv -E = \text{const} \ , \tag{24}$$

$$p_\phi = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi} \equiv L = \text{const} \ , \tag{25}$$

and

$$p_r = g_{rr}\dot{r} \ . \tag{26}$$

The Lagrangian is independent of both $t$ and $\phi$. This implies that $E$ and $L$ are constants of the motion. The Hamiltonian of the system is given by [2–4] $\mathcal{H} = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L}$, which implies

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 = \delta = \text{const} \ , \tag{27}$$

where $\delta = 0$ for null geodesics and $\delta = 1$ for timelike geodesics. Substituting Eqs. (24)-(26) into (27), one finds

$$\dot{r}^2 = E^2\Big[1+\frac{a^2-b^2}{r^2}+\frac{2M(a-b)^2}{r^3}\Big] \tag{28}$$

for null geodesics.

The requirement $\dot{r}^2 = 0$ for a null circular geodesic [2–4] yields

$$b_\pm = \frac{\sqrt{\Delta}\mp\frac{2Ma}{r}}{1-\frac{2M}{r}} \ , \tag{29}$$

which in view of Eq. (19) implies

$$b_\pm = \frac{T(r_\gamma)}{2\pi} = \Omega^{-1}(r_\gamma) \ , \tag{30}$$

where $\Omega \equiv \Omega_\infty$ is the angular frequency of the orbit as measured by asymptotic observers. The requirement $(\dot{r}^2)' = 0$ [2–4] yields the equation

$$r^2 - 3Mr \pm 2a\sqrt{Mr} = 0 \ . \tag{31}$$

It is straightforward to show that a solution of this equation is given by [2, 3, 10]:

$$r_{\gamma\pm} = 2M[1 + \cos[\frac{2}{3}\cos^{-1}(\mp a/M)]] \ . \tag{32}$$

Taking cognizance of Eq. (22) for the radius $r_{\text{fast}}$ of the fastest circular trajectory, one realizes that the null circular geodesic (32) around a Kerr black hole is characterized by the shortest possible orbital period as measured by asymptotic observers:

$$r_{\text{fast}} = r_\gamma \ . \tag{33}$$

The shortest possible orbital period around a Kerr black hole is obtained by substituting the optimal radius (22) back into Eq. (19). One then finds that $T_{\min} \equiv \min_r\{T(r; a/M)\}$ monotonically decreases as the ratio $a/M$ increases: The spherically symmetric Schwarzschild black hole is characterized by

$$T_{\min} = 6\sqrt{3}\pi M \tag{34}$$

[compare this with the naive (and weaker) bound (1) for spherical objects], while the maximally rotating Kerr black hole is characterized by

$$T_{\min} = 4\pi M \ . \tag{35}$$

## IV. SUMMARY

In summary, we have studied circular orbits around central black holes. In particular, we have analyzed the dependence of the orbital period on the radius of the circular trajectory. It was shown that the *null* circular geodesic is characterized by the *shortest* possible orbital period as measured by asymptotic observers. We therefore conclude that null circular geodesics provide the fastest way to circle black holes.

It was pointed out in [4] that intriguing physical phenomena could occur in curved spacetimes for which there is a timelike circular geodesic with an angular frequency which equals the angular frequency of the unstable null circular geodesic. For instance, this would raise the interesting possibility of exciting the black-hole quasinormal frequencies by orbiting particles, possibly leading to instabilities of the spacetime [4]. However, our results rule out this scenario for black-hole spacetimes with only one null circular geodesic. In particular, we have shown that the null circular geodesic is characterized by the largest angular frequency as measured by asymptotic observers – no other circular trajectory (be it a timelike circular geodesic or a non-geodesic orbit) can have a larger angular frequency. Thus, such spacetimes are characterized by

$$\Omega_{\text{timelike}} < \Omega_{\text{null}} \tag{36}$$

for all timelike circular geodesics.

Finally, following the result (35) for the shortest possible orbital period around a Kerr black hole as measured by asymptotic (flat-space) observers, it is tempting to conjecture a general lower bound on orbital periods around compact objects:

$$T_\infty \geq 4\pi M \ , \tag{37}$$

where $M$ is the mass of the central object.

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